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Classical limit of the hydrogen atom

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Abstract. A wave packet which travels on an elliptic trajectory is constructed for the hydrogen atom. This is achieved by mapping the Schrödinger equation for the hydrogen atom into the equation for a four-dimensional oscillator with a constraint. A set of coherent states for the constrained oscillator are then shown to have at high average energy the classical limit properties as obtained for planetary motion, to a good approximation.

1. Introduction

In 1926, Schrödinger (1926) constructed a set of states for the one-dimensional harmonic oscillator which travel along the classical trajectory as non-spreading localised wave packets. These states which are also minimum uncertainty states are now well known as coherent states. Schrödinger conjectured that similar constructions could be made for the hydrogen electron which he envisaged as 'wave groups' moving around 'highly quantised Kepler ellipses' and ended with the remarks that 'technical difficulties in the calculation are greater than in the specially simple case which we have treated here'.

Since then many attempts have been made to understand the quasi-classical behaviour of the hydrogen atom. Nieto and Simmons (1978), in a series of papers, developed a general formalism for the construction of minimum uncertainty coherent states for different potentials, both confining and non-confining in one and three dimensions. These states, except for the harmonic oscillator and harmonic oscillator with centripetal barrier potentials (both of which have equally spaced energy levels), spread, and also in the course of time the centres of gravity of these packets become detached from the corresponding classical particles. Other approaches (Brown 1973, Jordan-Maclay 1972, Snieder 1983) involved the superposition of appropriate hydrogenic wavefunctions so that the resultant packet travels in the classical orbit. But unfortunately these attempts involved educated guesses as to what the appropriate combinations should be and moreover failed to give in an analytical way the 'Keplerian ellipses' of Schrödinger.

In this paper we present a very natural construction for wave packets whose motion satisfies all the laws of Kepler. A well known mapping relates the Coulomb problem in three dimensions to a constrained oscillator in four dimensions whose coherent state can then be constructed in a standard way. In § 2 we describe the mapping and the solution of the Coulomb problem through a solution of the corresponding oscillator problem. In § 3 we present for the sake of completeness a brief discussion of the coherent states of the harmonic oscillator. We apply these techniques in § 4 to the hydrogen atom and construct the 'wave groups' which have the classical limit properties

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that Schrödinger may have wanted. The algebraic details involved in the derivation of the key results are relegated to the appendices.

2. Hydrogen atom as an oscillator

The Schrödinger equation for the hydrogen atom

$$\left[-\frac{\hbar^2}{2m}\nabla^2-\frac{k}{r}\right]\psi=E\psi \qquad E<0 \tag{1}$$

in units of the Bohr radius $a=\hbar^2/mk$, $\lambda=8mk/\hbar^2=8/a$, $\eta^4=-8mE/\hbar^2$, reduces to the form

$$\left[4\nabla^2+\frac{\lambda}{r}-\eta^4\right]\psi=0. \tag{2}$$

Let us introduce complex coordinates ζ_A and ζ_B given by

$$x+\mathrm{i}y\equiv 2\zeta_A\zeta_B^* \qquad x-\mathrm{i}y\equiv 2\zeta_A^*\zeta_B \qquad z\equiv\zeta_A^*\zeta_A-\zeta_B^*\zeta_B \tag{3}$$

or equivalently

$$r=\zeta_A^*\zeta_A+\zeta_B^*\zeta_B \qquad \theta=\cos^{-1}\left(\frac{|\zeta_A|^2-|\zeta_B|^2}{|\zeta_A|^2+|\zeta_B|^2}\right) \qquad \phi=(\text{phase of }\zeta_A-\text{phase of }\zeta_B). \tag{4}$$

This is the complex form (Cornish 1984) of the Kustaanheimo-Stiefel (1965) transformation developed by these authors to study the perturbations of planetary orbits in classical mechanics. The mapping was also simultaneously discovered by Ravndahl and Toyoda (1967) as a product of two successive mappings; first to parabolic coordinates and then again to rectangular coordinates with the introduction of an extra coordinate. Barut *et al* (1979) used this mapping in their researches on the quantum theory of infinite component fields.

Solving for ζ_A and ζ_B from (4) yields

$$\zeta_A=\sqrt{r}\cos(\theta/2)\,\mathrm{e}^{\mathrm{i}(\sigma+\phi)/2} \qquad \zeta_B=\sqrt{r}\sin(\theta/2)\,\mathrm{e}^{\mathrm{i}(\sigma-\phi)/2} \qquad \sigma\text{ arbitrary}. \tag{5}$$

The Laplacian is transformed as

$$r\nabla^2=\frac{\partial}{\partial\zeta_A^*}\frac{\partial}{\partial\zeta_A}+\frac{\partial}{\partial\zeta_B^*}\frac{\partial}{\partial\zeta_B} \tag{6}$$

so that the Schrödinger equation in the new variables becomes

$$\left[4\left(\frac{\partial}{\partial\zeta_A^*}\frac{\partial}{\partial\zeta_A}+\frac{\partial}{\partial\zeta_B^*}\frac{\partial}{\partial\zeta_B}\right)+\lambda-\eta^4(\zeta_A^*\zeta_A+\zeta_B^*\zeta_B)\right]\psi=0 \tag{7}$$

together with the constraint

$$\frac{\partial\psi}{\partial\sigma}=0\Rightarrow\left(\zeta_A^*\frac{\partial}{\partial\zeta_A^*}-\zeta_A\frac{\partial}{\partial\zeta_A}\right)\psi=-\left(\zeta_B^*\frac{\partial}{\partial\zeta_B^*}-\zeta_B\frac{\partial}{\partial\zeta_B}\right)\psi. \tag{8}$$

This condition (8) eliminates the dependence of the actual wavefunction $\Psi(r,\theta,\phi)$ on the auxiliary phase σ introduced through the mapping of the physical hydrogen atom problem in three dimensions into the oscillator in the four variables contained through the real and imaginary parts of ζ_A and ζ_B.

Let us introduce creation and annihilation operators

$$a_+ \equiv \frac{1}{\eta}\left(\frac{\partial}{\partial\zeta_A}+\frac{\eta^2}{2}\zeta_A^*\right) \qquad a_- \equiv \frac{1}{\eta}\left(\frac{\partial}{\partial\zeta_A^*}+\frac{\eta^2}{2}\zeta_A\right)$$

$$a_+^+ = \frac{1}{\eta}\left(-\frac{\partial}{\partial\zeta_A^*}+\frac{\eta^2}{2}\zeta_A\right) \qquad a_-^+ = \frac{1}{\eta}\left(-\frac{\partial}{\partial\zeta_A}+\frac{\eta^2}{2}\zeta_A^*\right) \tag{9}$$

and similarly $b_\pm$ and their Hermitian adjoints corresponding to ζ_B. These operators satisfy the canonical commutation relations

$$[a_i, a_j^+] = \delta_{ij} \qquad [a_i, a_j] = 0 = [a_i^+, a_j^+] \qquad i, j = +, - \tag{10}$$

and similarly for the b. The a operators, of course, commute with the b operators. In terms of these operators the Schrödinger equation (7) and the constraint condition (8) assume the form

$$(a_+^+a_+ + a_-^+a_- + b_+^+b_+ + b_-^+b_- + 2)\psi = (\lambda/2\eta^2)\psi \tag{11}$$

$$(a_+^+a_+ - a_-^+a_-)\psi = -(b_+^+b_+ - b_-^+b_-)\psi. \tag{12}$$

Equation (11) is formally identical to the Schrödinger equation for two two-dimensional harmonic oscillators while equation (12) asserts that the two oscillators should have equal and opposite angular momenta. Since the operators involved are of Hermitian quadratic form, they have non-negative eigenvalues and we have

$$(\lambda/2\eta^2) = (n_+ + n_- + m_+ + m_- + 2) \tag{13}$$

and

$$n_+ - n_- = m_- - m_+ \tag{14}$$

the latter condition being the consequence of the constraint. We therefore have $(\lambda/2\eta^2) = 2(n_+ + m_+ + 1)$, and recalling that $\lambda = 8mk/\hbar^2$ and $\eta^2 = (-8E/ka)^{1/2}$, we immediately arrive at the Rydberg formula

$$E = -(k/2a)/(n_+ + m_+ + 1)^2. \tag{15}$$

3. The coherent state

The coherent states introduced by Schrödinger and revived from obscurity by Glauber (1963) and Sudarshan (1963) are constructed as superpositions of the normalised energy eigenstates of the harmonic oscillator

$$|\alpha\rangle \equiv \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{16}$$

where the energy of the state $|n\rangle$, $E_n = (n+\frac{1}{2})\hbar\omega$, is the eigenvalue of the Hamiltonian $H = p^2/2m + m\omega^2x^2/2 = (a^+a+\frac{1}{2})\hbar\omega$, a and a^+ being the usual creation and annihilation operators defined by $a = (p - \mathrm{i}m\omega x)(2m\omega\hbar)^{-1/2}$ and $a^+ = (p + \mathrm{i}m\omega x)(2m\omega\hbar)^{-1/2}$. These states are eigenfunctions of the annihilation operator and can also be obtained from the vacuum state by a displacement

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{17}$$

$$|\alpha\rangle = \tilde{D}(\alpha)|0\rangle = \exp(\alpha a^+ - \alpha^* a)|0\rangle. \tag{18}$$

The coherent state constitutes a state of minimum uncertainty, namely

$$\Delta x\,\Delta p = \hbar/2. \tag{19}$$

The coherent state (16), though a non-stationary state, develops with time in a rather simple manner, because of the linear energy spectrum $E=(n+\frac{1}{2})\hbar\omega$ of the component oscillator states, and evolves at a later time into $|\alpha, t\rangle = |\alpha(t)\rangle$ where $\alpha(t)=\alpha\,\mathrm{e}^{-\mathrm{i}\omega t}$, and taking $\alpha \equiv \lambda\,\mathrm{e}^{-\mathrm{i}\theta}$ it follows that

$$\langle \alpha, t|\,x\,|\alpha, t\rangle = \left[2\lambda\left(\frac{\hbar}{2m\omega}\right)^{1/2}\right]\sin(\omega t+\theta). \tag{20}$$

Identifying the constant in square brackets with the amplitude, the expectation value of the displacement in the coherent state behaves like that for a classical oscillator. In this sense the coherent state is called a classical state. Moreover the finiteness of the classical amplitude demands that the classical limit be taken in the following way: $\hbar\to 0$, $|\alpha|\to\infty$ in such a way that $\sqrt{\hbar}\,|\alpha|\to$ finite.

4. Coherent state for the hydrogen atom

We have seen in § 2 that the Coulomb problem is formally equivalent to a constrained oscillator problem and in § 3 we have seen how to construct coherent states. We therefore obtain our unconstrained coherent state as a simultaneous eigenfunction of the commuting annihilation operators a_+, a_-, b_+ and b_- with the eigenvalues α_+, α_-, β_+ and β_- respectively:

$$|\alpha_+\alpha_-\beta_+\beta_-\rangle = \sum_{\substack{n_+,n_-\\ m_+,m_-}=0}^{\infty} \frac{\alpha_+^{n_+}\alpha_-^{n_-}\beta_+^{m_+}\beta_-^{m_-}}{(n_+!\,n_-!\,m_+!\,m_-!)^{1/2}}|n_+, n_-, m_+, m_-\rangle. \tag{21a}$$

The constraint $n_+ + m_+ = n_- + m_-$ is implemented (as discussed by Bhaumik *et al* (1976)) by writing

$$\alpha_\pm = |\alpha_\pm|\exp[\mp\mathrm{i}(\Delta_1-\sigma)] \qquad \beta_\pm = |\beta_\pm|\exp[\pm\mathrm{i}(\Delta_2+\sigma)] \tag{21b}$$

and the projected state satisfying the constraint as

$$|(\alpha_+\alpha_-\beta_+\beta_-)\rangle = N\int_0^{2\pi}(\mathrm{d}\sigma/2\pi)|\alpha_+\alpha_-\beta_+\beta_-\rangle \tag{22}$$

where N is a normalisation constant. If we substitute in equation (22) the expression for the unconstrained state from equation (21), the Kronecker delta arising out of the σ integration will guarantee that the constraint condition (14) is satisfied. The normalisation integral (as derived in appendix 1) is easily carried out to give

$$N = [\![I_0\{2[(|\alpha_+|^2+|\beta_+|^2)(|\alpha_-|^2+|\beta_-|^2)]^{1/2}\}]\!]^{-1/2} \tag{23}$$

where I_0 is the Bessel function of imaginary argument and of order zero.

We want the orbit to lie in the xy plane and to that end we will calculate the expectation value of z in the state given by equation (22) and equate it to zero

$$0=\langle z\rangle = \langle \zeta_A^*\zeta_A - \zeta_B^*\zeta_B\rangle = (1/\eta^2)\langle (a_-^\dagger + a_+)(a_- + a_+^\dagger) - (b_-^\dagger + b_+)(b_- + b_+^\dagger)\rangle. \tag{24}$$

To leading order (see appendix 1) in the α and β this yields

$$0=\langle z\rangle \simeq (2/\eta^2)(|\alpha_+||\alpha_-| - |\beta_+||\beta_-|)\left(1+\frac{(|\alpha_+||\alpha_-|+|\beta_+||\beta_-|}{[(|\alpha_+|^2+|\beta_+|^2)(|\alpha_-|^2+|\beta_-)^2)]^{1/2}}\right). \tag{25}$$

Thus to confine the orbit to the xy plane a convenient parametrisation is

$$|\alpha_+| = \rho \cos \chi \qquad |\alpha_-| = \mu \sin \chi \qquad |\beta_+| = \rho \sin \chi \qquad |\beta_-| = \mu \cos \chi \tag{26}$$

and we have for the normalised state

$$|\rho, \mu, \chi, \Delta_1, \Delta_2\rangle = [I_0(2\rho\mu)]^{-1/2} \int_0^{2\pi} (\mathrm{d}\sigma/2\pi) \times |\rho \cos \chi\, \mathrm{e}^{-\mathrm{i}(\Delta_1-\sigma)}, \mu \sin \chi\, \mathrm{e}^{\mathrm{i}(\Delta_1-\sigma)}, \rho \sin \chi\, \mathrm{e}^{\mathrm{i}(\Delta_2+\sigma)}, \mu \cos \chi\, \mathrm{e}^{-\mathrm{i}(\Delta_2+\sigma)}\rangle. \tag{27}$$

We shall see that all physically interesting quantities will depend on the product $\rho\mu$, χ and $\Delta_1 - \Delta_2$ and that these three parameters will be found to be related to the size, eccentricity and the orientation of the orbit.

Next we introduce the time evolution of the state. For this purpose we note that the coefficients of expansion in the state are peaked around the maximum of

$$(|\alpha_+|^{n_+}|\alpha_-|^{n_-}|\beta_+|^{m_+}|\beta_-|^{m_-})(n_+!\,n_-!\,m_+!\,m_-!)^{-1/2}$$

respecting the constraint $n_+ + m_+ = n_- + m_-$. Implementing the constraint by means of Lagrange multipliers and using Stirling's approximation for the factorials we see that the following values maximise the coefficient:

$$N_+ = (n_+)_{\max} = (\mu/\rho)|\alpha_+|^2 = \mu\rho \cos^2 \chi \qquad N_- = (n_-)_{\max} = (\rho/\mu)|\alpha_-|^2 = \mu\rho \sin^2 \chi$$
$$M_+ = (m_+)_{\max} = (\mu/\rho)|\beta_+|^2 = \mu\rho \sin^2 \chi \qquad M_- = (m_-)_{\max} = (\rho/\mu)|\beta_-|^2 = \mu\rho \cos^2 \chi. \tag{28}$$

We can therefore expand the energy in a power series around these maximum values. Thus

$$E = -(k/2a)(n_+ + m_+ + 1)^{-2} = E_{\text{peak}} + \hbar\omega_c(\delta_+ + \varepsilon_+ + \delta_- + \varepsilon_-) - \tfrac{3}{4}(\hbar\omega_c/\mu\rho)(\delta_+ + \varepsilon_+ + \delta_- + \varepsilon_-)^2 + \dots \tag{29}$$

where

$$\delta_\pm = n_\pm - N_\pm \qquad \varepsilon_\pm = m_\pm - M_\pm$$
$$\omega_c = (mk^2/2\hbar^3)(\mu\rho + 1)^{-3} \sim mk^2/2(\hbar\rho\mu)^3 \tag{30}$$

and

$$E_{\text{peak}} = -(2mk^2/\hbar^2)(N_+ + N_- + M_+ + M_- + 2)^{-2} = -(mk^2/2\hbar^2)(\mu\rho + 1)^{-2}. \tag{31}$$

The time evolution of the state entering through the factor $\mathrm{e}^{-\mathrm{i}Et/\hbar}$, induces, to leading order in $(1/\mu\rho)$, a simple time dependence of $\mathrm{e}^{-\mathrm{i}\omega_c t}$ to each of the α and β. Thus the time dependent state is given (up to a phase factor $\mathrm{e}^{-\mathrm{i}E_{\text{peak}}t/\hbar}$) by

$$|\rho\mu, \chi, \Delta_1, \Delta_2, t\rangle \simeq [I_0(2\rho\mu)]^{-1/2} \int_0^{2\pi} (d\sigma/2\pi) \times |\rho \cos \chi \exp[-\mathrm{i}\omega_c t - \mathrm{i}(\Delta_1 - \sigma)], \dots, \mu \sin \chi \exp[-\mathrm{i}\omega_c t + \mathrm{i}(\Delta_1 - \sigma)] \times \rho \sin \chi \exp[-\mathrm{i}\omega_c t + \mathrm{i}(\Delta_2 + \sigma)], \dots, \mu \cos \chi \exp[-\mathrm{i}\omega_c t - \mathrm{i}(\Delta_2 + \sigma)]\rangle. \tag{32}$$

With this time dependence for the state, the expectation value of z does not evolve with time. In other words, the orbit, once fixed in the xy plane, will remain so forever.

We can now calculate the expectation value of all dynamical variables expressing them in terms of creation and annihilation operators:

$$\langle x\rangle=(2\rho\mu/\eta^2)[\cos(\Delta_1-\Delta_2)(\cos 2\omega_c t+\sin 2\chi)+\sin(\Delta_1-\Delta_2)\sin 2\omega_c t\cos 2\chi]$$
$$\langle y\rangle=(2\rho\mu/\eta^2)[-\sin(\Delta_1-\Delta_2)(\cos 2\omega_c t+\sin 2\chi)+\cos(\Delta_1-\Delta_2)\sin 2\omega_c t\cos 2\chi]$$
$$\langle z\rangle=0=\langle L_x\rangle=\langle L_y\rangle \qquad \langle L_z\rangle=\rho\mu\hbar\cos 2\chi$$
$$\langle r\rangle=(2\rho\mu/\eta^2)(1+\sin 2\chi\cos 2\omega_c t). \tag{33}$$

The expressions for $\langle x\rangle$ and $\langle y\rangle$ become considerably simplified if we rotate the coordinate system about the z axis by an angle $-(\Delta_1-\Delta_2)$. The resulting values for $\langle x\rangle$ and $\langle y\rangle$ then satisfy the equation of an ellipse with its axes oriented along the coordinate axes. Thus $(\Delta_1-\Delta_2)$ is the angle made by the major axis, which is also the direction of the Runge–Lenz vector, with the x axis. Henceforth we will choose this angle to be zero:

$$\frac{[\langle x\rangle-(2\mu\rho/\eta^2)\sin 2\chi]^2}{(2\mu\rho/\eta^2)^2}+\frac{y^2}{[(2\mu\rho/\eta^2)\cos 2\chi]^2}=1. \tag{34}$$

The parameters of the orbit can now be easily read off:

$$\text{semi-major axis}=2\mu\rho/\eta^2=\mu^2\rho^2\hbar^2/mk^2 \tag{35}$$
$$\text{semi-minor axis}=2\mu\rho\cos 2\chi/\eta^2=(\mu^2\rho^2\hbar^2/mk^2)\cos 2\chi \tag{36}$$
$$\text{eccentricity } e=\sin 2\chi. \tag{37}$$

The energy of the state obtained as the expectation value of the Hamiltonian is given by

$$E=-(mk^2)/2\rho^2\mu^2\hbar^2.$$

Since $\langle x\rangle$ and $\langle y\rangle$ are periodic with frequency $2\omega_c$, the time period of revolution $T=2\pi/2\omega_c$ and Kepler's third law (Gerry 1984) follows:

$$T^2=\frac{\pi^2}{\omega_c^2}=\frac{4\pi^2(\hbar\rho\mu)^4}{k^4m^2}=\frac{4\pi m}{k}(\text{semi-major axis})^3. \tag{38}$$

The normalised wavefunction (see appendix 1) in the coordinate representation is found to be

$$\langle r\theta\phi|\rho\mu,\chi;t\rangle=(\eta^3/8\pi)\,\mathrm{e}^{-\eta^2r/2}[I_0(\rho\mu)+\rho\mu I_1(\rho\mu)]^{-1/2}$$
$$\times I_0[(2\eta^2\rho\mu r)^{1/2}\,\mathrm{e}^{\mathrm{i}\omega_c t}(\sin 2\chi+\sin\theta\cos\phi+\mathrm{i}\sin\theta\cos 2\chi\sin\phi)^{1/2}]. \tag{39}$$

For a circular orbit $\chi=0$ one can easily calculate the radial width of the wave packet

$$\Delta r=[\langle r^2\rangle-\langle r\rangle^2]^{1/2}=\tfrac{3}{2}[(\rho\mu\hbar^2/-Em)]^{1/2}=(3a/\sqrt{2})(L_z/\hbar)^{3/2} \tag{40}$$

which is time independent and vanishes in the classical limit $\hbar\to 0$.

We have, thus far, not considered the spreading of the wave packet with time. To understand, in a simple situation, how it disperses let us concentrate, for the moment, on a special state moving on a circular trajectory ($\chi=0$):

$$|\mu\rho,t\rangle=(\pi\mu\rho)^{-1/4}\int_{-\infty}^{+\infty}\mathrm{d}\nu\,\exp[-(\nu^2/2\mu\rho)-2\mathrm{i}\omega_c t\nu+3\mathrm{i}\omega_c t\nu^2/\mu\rho]$$
$$\times|\mu\rho+\nu,0,0,\mu\rho+\nu\rangle. \tag{41}$$

In the above we have retained the third term in the energy expansion (29) in the time evolution of the state and replaced the summation by an integration over the quantum numbers (for details see appendix 2) which can be carried out analytically. As before, $\langle r\rangle$ turns out to be $(2\mu\rho/\hbar^2)$ and $\Delta r(=3\hbar(\mu\rho/4(-E)m)^{1/2})$ is time independent. The plane of the orbit, on average, also remains fixed and has a fixed width in the z direction. Most interesting are the time evolutions of the average values of x and y

$$\langle x\rangle = (2/\eta^2)\exp[-(9\omega_c^2t^2)/(\mu\rho)-1/(4\mu\rho)](\mu\rho\cos 2\omega_c t+2\omega_c t\sin 2\omega_c t+\dots) \tag{42}$$

$$\langle y\rangle = (2/\eta^2)\exp[-(9\omega_c^2t^2)/(\mu\rho)-1/(4\mu\rho)](\mu\rho\sin 2\omega_c t-2\omega_c t\cos 2\omega_c t+\dots). \tag{43}$$

For large times both $\langle x\rangle$ and $\langle y\rangle$ go to zero keeping $\langle r\rangle$ constant. This simply means that the packet spreads along the trajectory and asymptotically becomes uniformly distributed over a ring. The terms linear in time need careful consideration. Such terms remind us of the secular terms that arise in the perturbative solution of non-linear differential equations, e.g. the equation for the Duffing oscillator (Nayfeh 1973) where a series of such terms when summed up again yield oscillatory functions. We anticipate a similar presence of secular terms here and identify the terms in the bracket as the first two terms in the expansion of

$$\langle x\rangle = (2\mu\rho/\eta^2)\exp[-(9\omega_c^2t^2)/(\mu\rho)-1/(4\mu\rho)]\cos[2\omega_c t-t\tan^{-1}(2\omega_c/\mu\rho)+\dots] \tag{44}$$

$$\langle y\rangle = (2\mu\rho/\eta^2)\exp[-(9\omega_c^2t^2)/(\mu\rho)-1/(4\mu\rho)]\sin[2\omega_c t-t\tan^{-1}(2\omega_c/\mu\rho)+\dots]. \tag{45}$$

Since the product $\mu\rho$ is related to the size of the orbit, the second terms in the arguments of the trigonometric functions give just an amplitude dependent frequency shift.

The calculation for the dispersion of the packet on an elliptic orbit can be carried out in an analogous manner. We give here only the conclusions. It turns out that an elliptic orbit is unstable against radial spreading in such a way that with time the orbit approaches a circular configuration keeping, of course, the energy and angular momentum conserved. Thus, asymptotically, the conclusions obtained for the circular orbit case apply to the elliptic orbit too; the probability distribution ultimately gets uniformly smeared over an asymptotic circular ring of finite, time independent width.

From equations (44) and (45) we see that the time $\tau_s=(\mu\rho)^{1/2}/3\omega_c$ plays the role of a characteristic time in which a wave packet gets smeared over a circular orbit. Let us make an estimate of this time for a macroscopic system like the Earth moving round the Sun. Eliminating $\mu\rho$ in terms of the orbit radius r we find

$$\tau_s=(2M/\hbar\omega_c)^{1/2}(r/3)$$

and taking $\hbar=10^{-27}$ erg s, $M=6\times10^{27}$ g, $r=1.5\times10^{13}$ cm and $T_{\text{Earth}}=1\text{ yr}\simeq\pi\times10^7$ s, the lifetime of the Earth as a compact object comes out to be

$$\tau_s\sim10^{36}\text{ yr}$$

which being considerably larger than the present age (4.57 billion yr) of the Solar System (Kirsten 1978) is comforting!

Appendix 1

Here we give a derivation of the normalisation integral. From equation (22) we find $1=\langle(\alpha_+\alpha_-\beta_+\beta_-)|(\alpha_+\alpha_-\beta_+\beta_-)\rangle$

$$=N^2\int_0^{2\pi}\int_0^{2\pi}(\mathrm{d}\sigma/2\pi)(\mathrm{d}\sigma'/2\pi)\langle\alpha'_+\alpha'_-\beta'_+\beta'_-|\alpha_+\alpha_-\beta_+\beta_-\rangle$$

where the primes indicate quantities defined in equation (21*b*) with $\sigma\to\sigma'$ necessary in view of the fact that coherent states do not constitute an orthogonal set. Inserting in the unconstrained state (which is not normalised), the expansion given by equation (21*a*) and using the parametrisation for the α and β from equations (21*b*) and invoking the orthonormality of the harmonic oscillator states we obtain

$$1=N^2\sum_{n_\pm m_\pm}\int\int(\mathrm{d}\sigma/2\pi)(\mathrm{d}\sigma'/2\pi)$$
$$\times[(|\alpha_+|^2)^{n_+}/(n_+!)][(|\alpha_-|^2)^{n_-}/(n_-!)][(|\beta_+|^2)^{m_+}/(m_+!)][(|\beta_-|^2)^{m_-}/(m_-!)]$$
$$\times\exp[\mathrm{i}(n_++m_+-n_--m_-)(\sigma-\sigma')]$$
$$=N^2\int\int(\mathrm{d}\sigma/2\pi)(\mathrm{d}\sigma'/2\pi)\exp[x\cos(\sigma-\sigma')+\mathrm{i}y\sin(\sigma-\sigma')]$$

where

$$x=|\alpha_+|^2+|\alpha_-|^2+|\beta_+|^2+|\beta_-|^2$$
$$y=|\alpha_+|^2+|\alpha_-|^2-|\beta_+|^2-|\beta_-|^2.$$

To arrive at the above we have carried out the summations and split the resulting exponentials into sines and cosines. Next we use the formulae

$$\mathrm{e}^{\mathrm{i}z\sin\theta}=\sum_k J_k(z)\,\mathrm{e}^{\mathrm{i}k\theta}\qquad \mathrm{e}^{z\cos\theta}=\sum_k J_k(-\mathrm{i}z)\,\mathrm{e}^{\mathrm{i}k[(\pi/2)-\theta]}$$

and carry out the σ integrations to yield

$$1=N^2\sum_k J_k(-\mathrm{i}x)J_k(y)\,\mathrm{e}^{\mathrm{i}k\pi/2}.$$

The summation over k is readily done with the help of the addition formulae for Bessel functions (Watson 1952) to give us the desired result, namely equation (22).

The calculation of the expectation value of z proceeds in an analogous manner. Since the unconstrained states are eigenstates of the annihilation operators we obtain from equation (24)

$$\langle z\rangle=(2/\eta^2)(|\alpha_+||\alpha_-|-|\beta_+||\beta_-|)+(2N^2/\eta^2)\int\int(\mathrm{d}\sigma/2\pi)(\mathrm{d}\sigma'/2\pi)$$
$$\times(|\alpha_+|^2\,\mathrm{e}^{\mathrm{i}(\sigma-\sigma')}+|\alpha_-|^2\,\mathrm{e}^{-\mathrm{i}(\sigma-\sigma')}-|\beta_+|^2\,\mathrm{e}^{\mathrm{i}(\sigma-\sigma')}$$
$$-|\beta_-|^2\,\mathrm{e}^{-\mathrm{i}(\sigma-\sigma')})\langle\alpha'_+\alpha'_-\beta'_+\beta'_-|\alpha_+\alpha_-\beta_+\beta_-\rangle$$

where again the prime indicates that the quantities involved are functions of σ'. In the above we have neglected terms which arise out of the re-ordering of the creation and annihilation operators, i.e. the commutators which, of course, are one order of magnitude lower in the α and β. The integration over the σ can now be carried out exactly as in the case of the normalisation integral to yield equation (25).

The coordinate representation (39) for the constrained normalised state is obtained by recognising that the unconstrained state (21a) is a simultaneous eigenfunction of the annihilation operators whose coordinate representation is given by equation (9). Solution of the eigenvalue equations, which are linear first order differential equations, gives us the unconstrained state

$$\langle\zeta_A\zeta_A^*\zeta_B\zeta_B^*|\alpha_+\alpha_-\beta_+\beta_-\rangle = \exp[-(\eta^2/2)(|\zeta_A|^2+|\zeta_B|^2)+\eta(\alpha_+\zeta_A+\alpha_-\zeta_A^*+\beta_+\zeta_B+\beta_-\zeta_B^*)].$$

The constrained coherent state $|(\alpha_+\alpha_-\beta_+\beta_-)\rangle$ is obtained from the above by using the parametrisations (21b) and (26) for the α and β and carrying out integration with respect to σ as in (22). We also explicitly institute a time dependence $e^{-i\omega_c t}$ for each of the α and β. The transition from the variables ζ_A and ζ_B to the spherical polar coordinates is obtained through (5). Finally equation (39) is arrived at after some algebraic manipulations involving the generating function and the addition theorem of Bessel functions.

Appendix 2

For the derivation of the special state representing circular trajectories with $\chi = 0$, we superpose harmonic oscillator states with $m_+ = n_- = 0$ so that the constraint equation (14) reduces to $n_+ = m_-$.

Thus we have

$$|\alpha_+\beta_-\rangle = \sum_{n_+=0}^{\infty} (\alpha_+\beta_-)^{n_+}/(n_+!)|n_+, 0, 0, m_- = n_+\rangle.$$

Using the parametrisation employed to confine the orbit to the xy plane (equation (26)), for circular orbits ($\chi = 0$) we have

$$|\alpha_+| = \rho \qquad |\beta_-| = \mu.$$

The coefficients of expansion in the state are peaked around $N_+ = (n_+)_{\max} = \mu\rho$. Expanding the coefficient in the power series around the maximum value and retaining terms quadratic in $\nu \equiv n_+ - \mu\rho$ we have

$$(|\alpha_+||\beta_+|)^{n_+}/(n_+!) \to \exp(\mu\rho - \nu^2/2\mu\rho).$$

The energy expansion (29) for this special state becomes

$$E = E_{\text{peak}} + \hbar\omega_c(2\nu) - (3\hbar\omega_c/\mu\rho)\nu^2 + \ldots$$
$$E_{\text{peak}} = -(mk^2/2\hbar^2)(\mu\rho+1)^{-2}.$$

Thus under these approximations we can write the special state under consideration at any instant of time t as (up to a phase factor $\exp(-iE_{\text{peak}}t/\hbar)$),

$$|\mu\rho, t\rangle = N \sum_{\nu=-\infty}^{+\infty} \exp[-(\nu^2/2\mu\rho) - 2i\omega_c t\nu + (3i\omega_c t\nu^2)/(\mu\rho) + \ldots]|\mu\rho+\nu, 0, 0, \mu\rho+\nu\rangle.$$

Here N is the normalisation constant.

Replacing the summation by an integration we arrive at (41)

$$|\mu\rho, t\rangle = N \int_{-\infty}^{+\infty} \mathrm{d}\nu \exp[-(\nu^2/2\mu\rho)$$

$$-2\mathrm{i}\omega_c t\nu + (3\mathrm{i}\omega_c t\nu^2)/(\mu\rho) + \ldots]\,|\mu\rho+\nu, 0, 0, \mu\rho+\nu\rangle.$$

The normalisation constant can easily be evaluated to obtain

$$N = (\pi\mu\rho)^{-1/4}.$$

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